

THE COMMUNICATION COMPLEXITY OF XOR GAMES VIA SUMMING OPERATORS

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ABSTRACT. The discrepancy method is widely used to find lower bounds for communication complexity of XOR games. It is well known that these bounds can be far from optimal. In this context Disjointness is usually mentioned as a case where the method fails to give good bounds, because the increment of the value of the game is linear (rather than exponential) in the number of communicated bits. We show in this paper the existence of XOR games where the discrepancy method yields bounds as poor as one desires. Indeed, we show the existence of such games with any previously prescribed value. Specifically we prove the following:

For any number of bits c and every $0 < \delta < 1$ and for every $\epsilon > 0$, we show the existence of a XOR game such that its value, both without communication or with the use of c bits of communication, is contained in the interval $(\delta - \epsilon, \delta + \epsilon)$.

To prove this result we apply the theory of p -summing operators, a central topic in Banach space theory. We show in the paper other applications of this theory to the study of the communication complexity of XOR games.

1. INTRODUCTION

A XOR game $G = (f, \pi)$ with N inputs on each side is defined by a function

$$f : [N] \times [N] \longrightarrow \{-1, 1\}$$

together with a probability distribution $\pi : [N] \times [N] \longrightarrow [0, 1]$. Alice and Bob receive as inputs $x, y \in [N]$ respectively and each of them must answer a number $a, b \in \{-1, 1\}$, so that $f(x, y) = a \cdot b$. They can also be viewed as linear combinations of the correlations achieved by the two parties when they are asked questions x, y .

XOR games are a very natural model for the study of communication complexity in computation as in Yao's model ([9]). They have also been used for the study of complexity classes ([21]), hardness of approximation ([8]), or for a better understanding of parallel repetition results, both in the classical and the quantum contexts ([12], [6], [18], [2])

In the context of quantum information, XOR games appear often with the name of correlation Bell inequalities. The so-called CHSH inequality has an extraordinary relevance

in this context [22]. They also provide an excellent testbed to study the relation between classical computation, quantum computation and communication complexity ([19], [1]).

The discrepancy method is one of the techniques most widely used to find lower bounds for the communication complexity of a XOR game. This method is known to give poor bounds in certain cases. We show in Theorem 1.1 that these bounds can be as poor as one wants.

We introduce the theory of p -summing operators with few vectors to study the communication complexity of XOR games. This leads us to the use of classical tools in the local theory of Banach spaces, like Grothendieck inequality, Chevet inequality, p -stable measures and the concentration of measure phenomenon. We must mention here the papers [16], [15] where techniques related to ours have also been used.

We say that a joint strategy between Alice and Bob γ is c -simulable if Alice and Bob can simulate γ using c -bits of communication. We denote these strategies by \mathcal{S}_c . We say that γ is c -simulable from Alice to Bob if they can simulate it when Alice sends c -bits of one way communication to Bob. We denote these strategies by \mathcal{S}_c^1 .

For a game G we define its values $\omega(G), \omega_c(G), \omega_c^1(G)$ as the maximum value that it attains on the strategies in $\mathcal{L}, \mathcal{S}_c, \mathcal{S}_c^1$ respectively.

The discrepancy method, as stated in [9, Proposition 3.28] tells us that, for every game G , $\omega_c(G) \leq 2^c \omega(G)$. Disjointness function is usually shown as an example where the discrepancy method fails to give good lower bounds for the communication complexity, since the increment $\frac{\omega_c(G)}{\omega(G)}$ is only linear in c .

In our main result, we show the existence of XOR games for which the performance of the discrepancy method is as poor as one desires. Specifically, for any prescribed $0 < \delta < 1$ and for any number of bits c we prove the existence of a XOR game G such that both $\omega(G)$ and $\omega_c(G)$ are as close to δ (and hence also to each other) as we want.

Theorem 1.1. *For every real number $0 < \delta < 1$, for every $c \in \mathbb{N}$ and for every $\epsilon > 0$ there exists a natural number N and a XOR game G with N inputs per player such that:*

$$\omega(G) \sim_\epsilon \delta \sim_\epsilon \omega_c(G),$$

where we use the notation $a \sim_\epsilon b$ to denote $b - \epsilon \leq a \leq b + \epsilon$ for every $a, b, \epsilon > 0$.

Next, we study how sharp the bound given by the discrepancy method is, taking also into account the number of inputs N . We do a full study for the case of one-way communication.

Our techniques can also be applied to more general cases. We use the notation \simeq to denote equality up to universal constants (independent of N and c).

Theorem 1.2. *For every XOR game G with N inputs per player, we have:*

a) $\omega_c^1(G) \leq K_G 2^{\frac{c}{2}} \omega(G)$ and

b) $\omega_c^1(G) \geq \frac{2^{\frac{c}{2}}}{K_G \sqrt{N}}.$

These inequalities are tight in the sense that there exist games J, H with N inputs per player such that

$$(1) \quad \omega_c^1(J) \simeq 2^{\frac{c}{2}} \omega(J) \text{ and}$$

$$(2) \quad \omega_c^1(H) \simeq \frac{2^{\frac{c}{2}}}{\sqrt{N}}.$$

Actually, in Proposition 4.1 below, we prove that, for big values of N , “most” games verify conditions (1) and (2).

The structure of the paper is the following: In Section 2 we introduce the notation and the formalism we will use. Next we introduce the mathematical tools that we need (p -summing operators with few vectors, Grothendieck inequality and Chevet inequality) and finally we state and prove the link between communication complexity and the theory of p -summing operators with few vectors.

Section 3 is devoted to the proof of Theorem 1.1, and Theorem 1.2 is proved in Section 4.

2. NOTATION AND MATHEMATICAL TOOLS

2.1. Notation. We will need the following notations and results from tensor product theory. Given a normed space X , we write X^* for its dual space with its natural dual norm and B_X for its closed unit ball. Given two normed spaces X, Y , and an element $u \in X \otimes Y$, we define its *projective* norm $\|u\|_\pi$ as

$$\|u\|_\pi = \inf \left\{ \sum_i \|x^i\| \|y^i\|, \text{ where } u = \sum_i x^i \otimes y^i \right\}$$

We write $X \otimes_\pi Y$ for the tensor product of X and Y endowed with the projective norm.

We can also define the *injective* norm of u as

$$\|u\|_\epsilon = \sup \left\{ \sum_i x^*(x^i) y^*(y^i), \text{ where } u = \sum_i x^i \otimes y^i, x^* \in B_{X^*}, y^* \in B_{Y^*} \right\}$$

and we write $X \otimes_\epsilon Y$ for their tensor product endowed with the injective norm.

In this note, X and Y will always be finite dimensional. It is well known (and not hard to see) that in that case $(X \otimes_\pi Y)^* = X^* \otimes_\epsilon Y^*$ and $(X \otimes_\epsilon Y)^* = X^* \otimes_\pi Y^*$.

In this paper we will always see games $(G_{x,y})_{x,y=1}^N = G$ as elements in $(\ell_\infty^N \otimes \ell_\infty^N)^*$, the algebraic dual of $\ell_\infty^N \otimes \ell_\infty^N$. We view the correlations attained by the players (or strategies) as elements $(\gamma_{x,y})_{x,y=1}^N = \gamma$ in $\ell_\infty^N \otimes \ell_\infty^N$. The value of the game G when the players play the strategy γ is

$$\langle G, \gamma \rangle = \sum_{x,y=1}^N G_{x,y} \gamma_{x,y}.$$

For an element $G \in (\ell_\infty^N \otimes \ell_\infty^N)^*$, we write $\|G\|_{op}$ for its norm as an element of $(\ell_\infty^N \otimes_\pi \ell_\infty^N)^*$, which coincides with its operator norm when we identify G with the operator $\tilde{G} : \ell_\infty^N \rightarrow \ell_1^N = (\ell_\infty^N)^*$ defined by $\tilde{G}(x)(y) = G(x, y)$. The way they are defined, XOR games are normalized in the sense that their norm as elements of $(\ell_\infty^N \otimes_\epsilon \ell_\infty^N)^*$ is always one.

When they do not communicate, Alice strategy upon receiving input x can be described as an element $\alpha(x, \lambda)$ in ℓ_∞^N , where λ stands for the state of their shared randomness. Similarly, Bob's strategy is $\beta(y, \lambda)$ so that their joint strategy is an element $(\gamma_{x,y})_{x,y=1}^N = \gamma = \sum_i \lambda_i \alpha^i \otimes \beta^i \in \ell_\infty^N \otimes \ell_\infty^N$ such that $\|\gamma\|_\pi \leq 1$. We will call these strategies *local* strategies and denote it by \mathcal{L} .

2.2. Summing operators. Following Grothendieck's work [7], the so called *local theory of Banach spaces* has been one of the cornerstones of modern functional analysis. Many of the main results in this theory can be expressed in terms of *summing operators*. We state next the definitions and results that we use in this paper. A detailed exposition in this area can be read, for instance, in [5].

Given a finite sequence with arbitrary length $(x_i)_{i=1}^n$ in a normed space X , and a real number $1 \leq p < \infty$, we define the *weakly p -summing* norm of $(x_i)_{i=1}^n$ by

$$\|(x_i)_{i=1}^n\|_p^w = \sup \left\{ \left(\sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{1}{p}}, \text{ where } x^* \in B_{X^*} \right\}.$$

Now, given an operator $T : X \rightarrow Y$ between normed spaces, we define its *p -summing* norm as

$$\pi_p(T) = \inf \{ C \text{ such that } \left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \|(x_i)_{i=1}^n\|_p^w \}$$

for every sequence $(x_i)_{i=1}^n \subset X$. It is well known that, for an operator $T : \ell_\infty^N \rightarrow Y$, $\pi_1(T) = \sum_{i=1}^N \|T(e_i)\|$.

If we fix $r \in \mathbb{N}$ and restrict the previous definition to sequences $(x_i)_{i=1}^r$ of maximum length r we obtain the definition of the p -summing with r vectors norm of T , which we denote by $\pi_p^r(T)$. Summing operators with few vectors have been studied by several authors, see for instance [20] and the references therein.

We will also use the following consequence of Grothendieck's inequality.

Theorem 2.1. *There exists an universal constant K_G such that, for any natural numbers N, M and every operator $u : \ell_\infty^N \longrightarrow \ell_1^M$,*

$$\pi_2(u) \leq K_G \|u\|.$$

2.3. Chevet inequality. The following result is known as Chevet inequality. It is usually stated for Gaussian random variables, we state it for Bernoulli random variables. See [14].

Theorem 2.2. *Given two normed spaces E, F , there exists a universal constant b such that*

$$\mathbb{E} \left\| \sum_{x,y} r_{x,y} \varphi_x \otimes \phi_y \right\|_{E \otimes_\epsilon F} \leq b \left(\|(\varphi_x)_x\|_2^w \mathbb{E} \left\| \sum_y r_y \phi_y \right\|_F + \|(\phi_y)_y\|_2^w \mathbb{E} \left\| \sum_x r_x \varphi_x \right\|_E \right),$$

where $r_x, r_y, r_{x,y}$ are independent Bernoulli random variables, and $(\varphi_x)_x, (\phi_y)_y$ are finite sequences in E, F . Actually, we can take $b = \sqrt{\frac{\pi}{2}}$ (see [3]).

We will apply Chevet inequality in the case $E = F = \ell_1^N$, and $\varphi_x = e_x, \phi_y = e_y, 1 \leq x, y \leq N$. In that case, we get easily

$$\|(e_x)_{x=1}^N\|_2^w = \sqrt{N} \text{ and } \mathbb{E} \left\| \sum_{y=1}^N r_y e_y \right\|_{\ell_1^N} = N$$

2.4. Communication complexity and p -summing operators. In this paper, we approach the study of the different types of strategies and the corresponding value of the games through tensor norms in $\ell_\infty^N \otimes \ell_\infty^N$ and its dual space. We have already mentioned that the local strategies can be identified with the norm unit ball of $\ell_\infty^N \otimes_\pi \ell_\infty^N$. It is easy to see that both \mathcal{S}_c^1 and \mathcal{S}_c are symmetric convex bodies of $\ell_\infty^N \otimes \ell_\infty^N$ with non empty interior. Hence, they define norms on this space, and therefore the value of a game G on them can be seen as the corresponding dual norm of the game.

Lemma 2.4 below is the starting point of our approach: It identifies the value of a game on the strategies in \mathcal{S}_c^1 as certain operator norm. We isolate the technical parts of the proof in the following lemma. Its proof follows from [5, Proposition 2.2 and Lemma 16.13].

Lemma 2.3. *Let $B = \{(\alpha_i)_{i=1}^r \subset \ell_\infty^N \text{ such that } \|(\alpha_i)_{i=1}^r\|_1^w \leq 1\}$. Given $A \subset [N]$, let $\alpha_A \in \ell_\infty^N$ be the element defined by $\alpha_A(x) = 1$ if $i \in A$ and $\alpha_A(x) = 0$ otherwise. Let $A = \{(\alpha_{A_i})_{i=1}^r; \text{ where } A_1, \dots, A_r \text{ is a partition of } [N]\}$. Then B is the symmetric convex hull of the elements in A .*

Lemma 2.4. *Given a game $(G_{x,y})_{x,y=1}^N$, $\omega(G) = \|\tilde{G}\|_{op}$ and $\omega_c^1(G) = \pi_1^{2^c}(\tilde{G})$.*

Proof. The first statement follows immediately by duality from the characterization of the local strategies.

We prove the second statement. Let us first see that $\omega_c^1(G) \leq \pi_1^{2^c}(\tilde{G})$. We assume that the communication Alice sends might be dependent on a variable $\lambda \in \Lambda$. We call $T(x, \lambda)$ to the word that Alice sends when she receives the input x and the random variable takes the value λ . We have that $1 \leq T(x, \lambda) \leq 2^c$. For fixed $1 \leq i \leq 2^c$ and $1 \leq x \leq N$, call

$$\Lambda_{i,x} = \{\lambda \in \Lambda \text{ such that } T(x, \lambda) = i\}.$$

For fixed i, λ , call

$$X_{i,\lambda} = \{x \text{ such that } T(x, \lambda) = i\}.$$

Calling α, β to the strategies followed by Alice and Bob, we have

$$\begin{aligned} \omega_c^1(G) &= \sup \sum_{x,y} G_{xy} \int_{\Lambda} \alpha(x, \lambda) \beta(y, \lambda, T(x, \lambda)) d\lambda = \\ &= \sup \sum_{x,y} G_{xy} \sum_i \int_{\Lambda_{i,x}} \alpha(x, \lambda) \beta(y, \lambda, T(x, \lambda)) d\lambda = \sum_{x,y} G_{xy} \sum_i \int_{\Lambda_{i,x}} \alpha(x, \lambda) \beta(y, \lambda, i) d\lambda = \\ &= \sum_{x,y} G_{xy} \sum_i \int_{\Lambda_{i,x}} \alpha(x, \lambda) \beta_i(y, \lambda) d\lambda = \int_{\Lambda} \sum_i \sum_{x \in X_{i,\lambda}} \sum_y G_{xy} \alpha(x, \lambda) \beta_i(y, \lambda) d\lambda. \end{aligned}$$

For fixed λ , $\sum_i \sum_{x \in X_{i,\lambda}} \sum_y G_{xy} \alpha(x, \lambda) \beta_i(y, \lambda)$ is bounded above by $\pi_1^{2^c}(\tilde{G})$ (use Lemma 2.3 for this). Considering the convex hull will not change this fact.

The reverse inequality follows easily from Lemma 2.3 and convexity. \square

We mentioned in the Section 2 that all XOR games with N inputs per player G have norm one considered as elements of $(\ell_\infty^N \otimes_\epsilon \ell_\infty^N)^*$. It is well known ([5]) that this is equivalent to the condition $\pi_1(\tilde{G}) = \pi_1^N(\tilde{G}) = 1$. In particular, if $c \geq \log N$, then $\omega_c^1(G) = \omega_c(G) = 1$.

3. PROOF OF THEOREM 1.1

Theorem 1.1 follows from Theorem 3.1 and Proposition 3.3 below.

Theorem 3.1. *For any real number $\alpha > 1$, positive integer t and $\epsilon > 0$, there exists a natural number N and an operator $T : \ell_\infty^N \rightarrow \ell_1^N$ such that*

- 1) $\|T\|_{op} \sim_\epsilon 1$
- 2) $\pi_1^t(T) \sim_\epsilon 1$
- 3) $\pi_1(T) \sim_\epsilon \alpha$

The game G that we look for in Theorem 1.1 is nothing but the game whose associated operator is $\tilde{G} = \frac{T}{\pi_1(T)}$, with the proper choices of ϵ , t and α .

The key point of the proof of Theorem 3.1 is Levi's embedding theorem, which says that, for every $1 < p < 2$, we have an isometric embedding of ℓ_p into $L_1[0, 1]$. Actually, the result is much more general (see [17] and [11]). This embedding is highly non-trivial and it is based on p -stable measures. We are interested in the $(1 + \epsilon)$ -isomorphic finite dimensional version of the theorem. Specifically, we use the following improvement of Levi's embedding theorem due to Johnson and Schechtman.

Theorem 3.2 (Theorem 1, [10]). *Let $\epsilon > 0$, and suppose that $0 < r < s < 2$ with $r \leq 1$. Then there exists $\beta = \beta(\epsilon, r, s) > 0$ so that if m and n are positive integers with $m \leq \beta n$, then ℓ_s^m is $(1 + \epsilon)$ -isomorphic to a subspace of ℓ_r^n .*

Note that, in the particular case of $r = 1$ and $1 < p < 2$, Theorem 3.2 assures the existence of $\beta = \beta(\epsilon, 1, p) > 0$ and an isomorphism $A : \ell_p^m \hookrightarrow \ell_1^n$ such that

$$(1 - \tau)\|x\|_{\ell_p^m} \leq \|Ax\|_{\ell_1^n} \leq (1 + \tau)\|x\|_{\ell_p^m}$$

for every $x \in \ell_p^m$.

Proof of Theorem 3.1. Let α, t and ϵ be as in the statement. We define:

$$\begin{aligned} \theta_0 &= \log(\alpha), \\ m_0 &= \min\{m \in \mathbb{N} : t^{\frac{\theta_0}{m}} < 1 + \epsilon\}, \\ k &= 2^{m_0} \text{ and} \\ q &= \frac{m_0}{\theta_0}. \end{aligned}$$

Note that we can assume that $2 < q < \infty$. Indeed, if it is not, we only have to consider a high enough m_0 . Then, we define p by $\frac{1}{p} + \frac{1}{q} = 1$ (so $1 < p < 2$). And we will denote $q = p'$. Note that, $t^{\frac{1}{p'}} < 1 + \epsilon$ and $k^{\frac{1}{p'}} = (2^{m_0})^{\frac{\theta_0}{m_0}} = 2^{\theta_0} = \alpha$.

We begin by considering the operator

$$S := k^{-\frac{1}{p}} \text{id} : \ell_\infty^k \rightarrow \ell_p^k.$$

It is not difficult to check that $\|S\| = \pi_p(S) = 1$ and $\pi_1(S) = k^{\frac{1}{p'}} = \alpha$ (see for instance [3]).

Now, we define the operator

$$T := A \circ S \circ P : \ell_\infty^N \rightarrow \ell_\infty^k \rightarrow \ell_p^k \hookrightarrow \ell_1^N,$$

where $N = \frac{k}{\beta}$ for the $\beta = \beta(\epsilon, 1, p)$ given by Theorem 3.2, A is the associated $1 + \epsilon$ -isomorphism given by the same theorem and $P : \ell_\infty^N \rightarrow \ell_\infty^k$ denotes the standard projection. Now, by Theorem 3.2 and the injectivity property of the p -summing operators (see for instance [3]), we know that $\|T\| \sim_\epsilon 1$, $\pi_p(T) \sim_\epsilon 1$ and $\pi_1(T) \sim_\epsilon \alpha$. We finish the proof if we show that $\pi_1^t(T) \sim_\epsilon 1$.

To see this, consider a sequence $x_1, \dots, x_t \in \ell_\infty^N$ such that

$$\sup\left\{\sum_{i=1}^t |x^*(x_i)| : x^* \in B_{\ell_1^N}\right\} \leq 1.$$

Then,

$$\sum_{i=1}^t \|T(x_i)\| \leq t^{\frac{1}{p'}} \left(\sum_{i=1}^t \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq (1 + \epsilon)^2.$$

A suitable adjust of the ϵ 's finishes the proof. \square

This result yields immediately a “one-way communication” version of Theorem 1.1. For the general version, we need the following simple result.

Proposition 3.3. *Let G be a XOR game and let c be a natural number. Then*

$$\omega_c(G) \leq 2^c \omega_c^1(G).$$

Proof. Applying convexity, we know that there exists a partition R_1, \dots, R_{2^c} of $[N] \times [N]$ in rectangles and sign vectors $(\alpha^i(x))_{x=1}^N, (\beta^i(y))_{y=1}^N$, with $\alpha^i(x) = \pm 1 = \beta^i(y)$ for every x, y, i such that

$$\omega_c(G) = \sum_{i=1}^{2^c} \sum_{x,y \in R_i} \alpha^i(x) \beta^i(y) M_{x,y}.$$

For every fixed $1 \leq x \leq N$ we define $i(x, y)$ as the unique i such that $(x, y) \in R_i$ and we consider the row of signs $(\alpha^{i(x,1)}(x) \beta^{i(x,1)}(1), \dots, \alpha^{i(x,N)}(x) \beta^{i(x,N)}(N))$. It is easy to see that there are at most 2^{2^c} different such rows. Clearly, 2^c bits suffice Alice to tell Bob which is the row associated to x . \square

4. PROOF OF THEOREM 1.2

Proof. a) Let G be a XOR game with N inputs per player, and let $\tilde{G} : \ell_\infty^N \rightarrow \ell_1^N$ be its associated operator, as in the introduction. Grothendieck's Theorem tells us that $\pi_2(\tilde{G}) \leq$

$K_G \|\tilde{G}\|_{op}$. Now, let $x_1, \dots, x_{2^c} \in \ell_\infty^N$ be a finite sequence such that $\|(x_i)_{i=1}^{2^c}\|_1^w \leq 1$. Then,

$$\omega_c^1(G) = \pi_1^{2^c}(\tilde{G}) \leq \sum_{i=1}^{2^c} \|\tilde{G}(x_i)\| \leq 2^{\frac{c}{2}} \left(\sum_{i=1}^{2^c} \|\tilde{G}(x_i)\|^2 \right)^{\frac{1}{2}} \leq 2^{\frac{c}{2}} K_G \|\tilde{G}\|_{op} = 2^{\frac{c}{2}} K_G \omega(G).$$

Let us see the optimality. Recall that we view games as elements in $(\ell_\infty^N \otimes \ell_\infty^N)^* = \ell_1^N \otimes \ell_1^N$. We apply Chevet inequality, to find a choice of signs $(\varepsilon_{x,y})_{x,y=1}^N$ such that

$$\begin{aligned} \left\| \sum_{x,y=1}^{2^c} \varepsilon_{x,y} e_x \otimes e_y \right\|_{\ell_1^{2^c} \otimes \ell_1^{2^c}} &\leq 1 \text{ and} \\ \left\| \sum_{x,y=1}^{2^c} \varepsilon_{x,y} e_x \otimes e_y \right\|_{\ell_1^{2^c} \otimes \pi \ell_1^{2^c}} &\succeq \sqrt{2^c}, \end{aligned}$$

where \succeq denotes inequality up to an universal constant. This defines an operator $T : \ell_\infty^{2^c} \rightarrow \ell_1^{2^c}$ such that $\|T\|_{op} \leq 1$ and $\pi_1(T) = \pi_1^{2^c}(T) \succeq \sqrt{2^c}$. We define $T' = \frac{T}{\pi_1(T)}$. Let now $P : \ell_\infty^N \rightarrow \ell_\infty^{2^c}$ be the canonical projection onto the first 2^c coordinates, and let $\varphi : \ell_1^{2^c} \rightarrow \ell_1^N$ be the canonical inclusion into the first 2^c coordinates. Then the game J defined by $\tilde{J} : \varphi \circ T' \circ P : \ell_\infty^N \rightarrow \ell_\infty^{2^c} \rightarrow \ell_1^{2^c} \rightarrow \ell_1^N$ verifies what we wanted.

b) Let G be as in the hypothesis. First we assume that $\frac{N}{2^c} = h \in \mathbb{N}$. Call A_j to the isometric copy of $\ell_\infty^{\frac{N}{2^c}}$ contained naturally in ℓ_∞^N considering only the basis elements e_i , with $(j-1)\frac{N}{2^c} < i \leq j\frac{N}{2^c}$. Then

$$\begin{aligned} 1 &= \pi_1(\tilde{G}) = \sum_{i=1}^N \|\tilde{G}(e_i)\| = \sum_{j=1}^{2^c} \sum_{i=1}^{\frac{N}{2^c}} \|\tilde{G}(e_{(j-1)\frac{N}{2^c}+i})\| \leq \\ &\leq \sum_{j=1}^{2^c} K_G \sqrt{\frac{N}{2^c}} \|\tilde{G}|_{A_j}\|_{op} \leq K_G \sqrt{\frac{N}{2^c}} \pi_1^{2^c}(\tilde{G}) = K_G \sqrt{\frac{N}{2^c}} \omega_c^1(G). \end{aligned}$$

Now we consider the case when $\frac{N}{2^c}$ is not an integer, and we denote p the smallest natural number such that $\frac{N}{2^c} \leq p$. Then again we have $\pi_1(G) = \pi_1^N(G) = \pi_1^{p2^c}(G) \leq \sqrt{p} K_G \pi_1^{2^c}(G) \leq 2 K_G \sqrt{\frac{N}{2^c}} \pi_1^{2^c}(G)$ and the result follows.

We see now the optimality of this result. Apply again Chevet inequality to find a choice of signs $(\varepsilon_{x,y})_{x,y=1}^N$ such that $\left\| \sum_{x,y=1}^N \varepsilon_{x,y} e_x \otimes e_y \right\|_{\ell_1^N \otimes \ell_1^N} \leq 1$ and $\left\| \sum_{x,y=1}^N \varepsilon_{x,y} e_x \otimes e_y \right\|_{\ell_1^N \otimes \pi \ell_1^N} \succeq \sqrt{N}$. Let $G' : \ell_\infty^N \rightarrow \ell_1^N$ be its associated operator and let G be the game associated to $\frac{G'}{\pi_1(G')}$. By a), we know that $\omega_c^1(G) \preceq \frac{2^{\frac{c}{2}}}{\sqrt{N}}$. \square

Actually, we can see that, for big values of N , “most” games essentially attain the bounds given above. We write the statement for the case of games $G = (f, u)$ with u the uniform

distribution. Similar results can be proved for other distributions. The tool now is the Concentration of Measure Phenomenon.

Proposition 4.1. *Let X_N be the set of games with N inputs per player defined by $G = (f, u)$, with u the uniform distribution. Consider in X_N the probability $\mu : \mathcal{P}(X_N) \rightarrow [0, 1]$ defined by $\mu(A) = \frac{\text{Card}(A)}{2^{N^2}}$. Let $r > 0$. If m is a median of $\omega(G)$ under μ , then*

$$\mu(\{G \text{ such that } |\omega(G) - m| \geq r\}) \leq 2e^{-\frac{N^2 r^2}{2}}.$$

Proof. The proof follows immediately from [13, Proposition 1.3] once we check that $\omega(G)$ is a 2-Lipschitz function under the normalized Hamming distance in X_N . \square

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